

Intermittency in Inhomogeneous Coupled Map Lattices

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We study the phenomenon of intermittency in an inhomogeneous lattice of coupled maps where the inhomogeneity appears in the form of different values of the map parameter at adjacent sites. This system exhibits spatio-temporal intermittency over substantial regions of parameter space. We also observed the unusual feature of purely spatial intermittency accompanied by temporal periodicity in some regions of the parameter space. The intermittency appears as a result of bifurcations of co-dimension two in such systems. We identify the types of bifurcations which lead to spatio-temporal or purely spatial intermittency and identify the mechanism of intermittency for one such case, that arising from the bifurcation of the synchronised fixed point solution, by examining the spatial return map. The intermittency near the bifurcation points is associated with power-law distributions for the laminar lengths. The scaling laws for the laminar length distributions are obtained. Three distinct types of intermittency characterised by power-laws with three exponents which fall in three distinct ranges can be seen. The structure associated with the tangent-period-doubling bifurcation undergoes further bifurcation to a spatio-temporally intermittent regime with an associated power-law exponent. Three of the exponents seen in this model show very good agreement with those observed in fluid experiments involving quasi-one dimensional geometries.

The existence of spatio-temporal as well as temporal intermittency is an interesting phenomenon which is found in a wide range of dynamical systems of physical interest such as oscillators [1], chemical reactions [2], pattern formation of various kinds [3], turbulence [4], fluid flows [5] and signals of all kinds. The phenomenon of temporal intermittency has been studied extensively and is relatively well understood [6]. Pomeau and Manneville [7] first established the route to chaos via temporal intermittency and studied its behaviour. The nature of intermittency in spatially extended dynamical systems, however has not been understood very well. The presence of spatial as well as temporal intermittency has implications for understanding the physics of pattern formation and for understanding the ubiquitous presence of structures in chaotic systems.

Coupled map lattices are particularly simple paradigms for systems with extended spatial dimension which show a wide range of interesting behaviour [8]. Spatio-temporal intermittency, which has been defined as a ‘fluctuating mixture of regular and turbulent domains [9]’, has been observed in such systems [8] [9] [10]. However, the mechanism by which such intermittency occurs has not been very clear. We study the phenomenon of spatio-temporal intermittency in an inhomogeneous lattice of coupled maps where the inhomogeneity appears in the form of different values of map parameter at different sites. Such lattices have been considered in the case of pinning studies [11] and in the context of control of spatio-temporal chaos [10]. The coupling between maps is diffusive and nearest neighbour. We consider the simplest case, that is, a situation where adjacent lattice sites have different values of the parameter and alternate sites have the same value of the parameter.

Our model is defined by the evolution equations

$$x_i^{t+1} = (1 - \epsilon)f(x_i^t, \mu) + \frac{\epsilon}{2}(f(x_{i-1}^t, \mu') + f(x_{i+1}^t, \mu')) \quad (1)$$

where $f(x) = \mu x(1 - x)$ is the logistic map and $\mu, \mu' \in [0, 4]$, x_i^t is the value of the variable x at the lattice site i at time t , and $0 \leq x \leq 1$. We set $\mu' = \mu - \gamma$ and use periodic boundary conditions where $2N$, the number of lattice sites is even. The synchronised fixed points of the system are given by $x = 0$ and $x = \frac{\mu - \gamma\epsilon - 1}{\mu - \gamma\epsilon}$

Expanding the evolution equations about the synchronised fixed point, the linear stability matrix has the form

$$J = \begin{bmatrix} (1 - \epsilon)f'_\mu(x) & \frac{\epsilon}{2}f'_{\mu'}(x) & 0 & 0 & \dots & 0 & \frac{\epsilon}{2}f'_{\mu'}(x) \\ \frac{\epsilon}{2}f'_\mu(x) & (1 - \epsilon)f'_{\mu'}(x) & \frac{\epsilon}{2}f'_\mu(x) & 0 & \dots & 0 & 0 \\ 0 & \frac{\epsilon}{2}f'_{\mu'}(x) & (1 - \epsilon)f'_\mu(x) & \frac{\epsilon}{2}f'_{\mu'}(x) & \dots & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{2}f'_{\mu'}(x) & (1 - \epsilon)f'_\mu(x) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & (1 - \epsilon)f'_\mu(x) & \frac{\epsilon}{2}f'_{\mu'}(x) \\ \frac{\epsilon}{2}f'_\mu(x) & 0 & \dots & \dots & \dots & \frac{\epsilon}{2}f'_{\mu'}(x) & (1 - \epsilon)f'_{\mu'}(x) \end{bmatrix} \quad (2)$$

This matrix is of the block circulant form. Thus it can be put in a block diagonal form by a similarity transformation [12], [13] of blocks $M(\theta)$ where $M(\theta)$ are 2×2 matrices of the form

$$M(\theta) = \begin{bmatrix} (1 - \epsilon)f'_\mu(x) & \frac{\epsilon}{2}(1 + e^{i\theta})f'_{\mu'}(x) \\ \frac{\epsilon}{2}(1 + e^{-i\theta})f'_\mu(x) & (1 - \epsilon)f'_{\mu'}(x) \end{bmatrix} \quad (3)$$

where, $\theta = \frac{2\pi(l-1)}{N}$, $l = 1, \dots, N$.

The eigenvalues of this matrix as a function of θ are given by

$$\lambda = \frac{(1 - \epsilon)(f'_\mu + f'_{\mu'}) \pm ((1 - \epsilon)(f'_\mu + f'_{\mu'})^2 - 2(1 - \epsilon)\epsilon(f'_\mu f'_{\mu'})^2(1 + \cos(\theta)))^{1/2}}{2} \quad (4)$$

It is clear from Eq. 4 that the eigenvalues of the matrices $M(\theta)$ are bounded between the largest eigen-value of the matrix $M(0)$ and the smallest eigen-value of the matrix $M(\pi)$.

For the $x = 0$ fixed point, the eigenvalue equations are

$$\begin{aligned}\lambda^2 - \lambda(1-\epsilon)(2\mu-\gamma) + \mu(1-2\epsilon)(\mu-\gamma) &= 0 && \text{for } M(0) \\ \lambda^2 - \lambda(1-\epsilon)(2\mu-\gamma) + \mu(1-\epsilon)^2(\mu-\gamma) &= 0 && \text{for } M(\pi)\end{aligned}\quad (5)$$

For the fixed point, $x = \frac{\mu-\gamma\epsilon-1}{\mu-\gamma\epsilon}$ the eigenvalue equations are

$$\begin{aligned}\lambda^2 - \lambda(1-\epsilon)(2\mu-\gamma)\xi + \mu(1-2\epsilon)(\mu-\gamma)\xi^2 &= 0 && \text{for } M(0) \\ \lambda^2 - \lambda(1-\epsilon)(2\mu-\gamma)\xi + \mu(1-\epsilon)^2\xi^2(\mu-\gamma) &= 0 && \text{for } M(\pi)\end{aligned}\quad (6)$$

where $\xi = \frac{\gamma\epsilon-\mu-2}{\mu-\gamma\epsilon}$.

We fix the value of μ at 4.0. The system now has two parameters ϵ and γ . It is clear from Eq. 4 that the eigen-values of the linear stability matrix are functions of ϵ and γ .

It is well-known [14] that bifurcations occur and the nature of the stable solution changes at the values of the parameters where the modulus of the eigen-values of the linear stability matrix crosses the unit circle. In the co-dimension one case, the tangent bifurcation can be seen where the largest real eigen-value crosses $+1$, the period-doubling bifurcation can be seen where the smallest real eigen-value crosses -1 , and a Hopf bifurcation can be seen where the complex eigen-values cross the unit circle. For the present system, the conditions for the eigenvalues to cross the unit circle define a set of curves in the two-parameter $\epsilon - \gamma$ space as listed in Table I. These curves intersect in several places where the equations are simultaneously satisfied. At these intersections two eigen-directions become unstable, resulting in bifurcations of co-dimension two. A rich variety of spatio-temporal behaviour can be seen in the neighbourhood of such points. We concentrate on regions which exhibit spatio-temporal intermittency where a fluctuating mixture of regular and irregular domains can be seen. Such solutions are possible over a large region of parameter space. Table I lists the possible bifurcations associated with intermittent behaviour and the different types of intermittent behaviour observed near such points. It is interesting to note that in addition to the phenomenon of spatio-temporal intermittency, the inhomogeneous lattice has regions in parameter space where pure spatial intermittency accompanied by temporally periodic behaviour can be seen i.e. the temporal behaviour is periodic, but spatially laminar and turbulent regions co-exist at any point in time (See Fig. 1). The temporally periodic spatial structure seen shows the unusual occurrence of long range spatial correlations. This phenomenon can be seen in the vicinity of the tangent-period doubling bifurcation whereas all the other bifurcations result in spatio-temporal intermittency. The bifurcations in case of the Double Hopf and period-doubling are inverse bifurcations as can be confirmed by a normal form analysis. The details of this calculation can be found elsewhere [19].

Table I shows that spatio-temporal as well as purely spatial intermittency can arise in the neighbourhood of co-dimension 2 bifurcations. We examine one such case, that of the tangent-period doubling bifurcation where the interesting case of purely spatial intermittency can be seen. Let $\chi_0(\gamma, \epsilon, \lambda)$ be the characteristic polynomial for $M(0)$ and $\chi_\pi(\gamma, \epsilon, \lambda)$ be the characteristic polynomial for $M(\pi)$. Then the conditions to be simultaneously satisfied for the Tangent-Period doubling bifurcation are

$$\begin{aligned}1 - (1-\epsilon)(2\mu-\gamma)\xi + \mu(1-2\epsilon)(\mu-\gamma)\xi^2 &= 0 \\ 1 + (1-\epsilon)(2\mu-\gamma)\xi + \mu(1-\epsilon)^2(\mu-\gamma)\xi^2 &= 0\end{aligned}\quad (7)$$

where $\xi = \frac{\gamma\epsilon-\mu-2}{\mu-\gamma\epsilon}$ as before. These conditions are simultaneously satisfied when $\gamma = 1.18$, $\epsilon = 0.63$.

We show the spatial intermittency present in the vicinity of this point in Fig 1(a). The size of lattice chosen was 1000 and it was iterated for 10,000 iterations after discarding

20,000 transients. The ordinate is the value of x at the i th site and abscissa is the site label. The lattice is plotted at a fixed time. In Fig.1(b) we plot the time evolution of the lattice which shows stable period two oscillations. Thus the intermittency is purely spatial in nature and temporally we have stable periodic behaviour. The temporally periodic spatial structure seen shows the unusual occurrence of long range spatial correlations. This can be seen in the distribution of laminar lengths which we will discuss shortly.

Another tool that can give some insight into the intermittency for this Tangent-Period-doubling case is the spatial return map plotted in Figure 3. Before the bifurcation, the return map shows only a single point corresponding to the synchronised fixed point solution. After the bifurcation a loop is seen. This is because intermediate values between the two period doubled points are also possible now. This loop acts as a relaminarisation mechanism and gives rise to intermittency. The region in which the intermittency occurs in inhomogeneous lattices is considerably bigger than that seen for homogeneous lattices. This is because the lattice is no longer symmetric under the "flip". Thus the periodic solutions which were stable for homogeneous lattices bifurcate and give rise to intermittency.

Each type of bifurcation gives rise to intermittency of a distinct type as typified by the distribution of laminar lengths. The distribution of laminar lengths shows power law behaviour, and the tangent, the period-doubling and the Hopf bifurcation are associated with their own distinct power laws. The distributions for the tangent-Hopf and tangent-period-doubling bifurcations scale with the power-laws associated with the Hopf and period-doubling bifurcations respectively. These power-laws are seen in the vicinity of the co-dimension two points listed in Table I and also in the neighbourhood of the lines which correspond to the tangent bifurcations. The question of whether spatio-temporal intermittency persists beyond these regions and whether the distribution of laminar lengths in the spatio-temporally intermittent regime crosses over from power-law behaviour to exponential behaviour is presently under investigation.

To distinguish between various kinds of intermittency, we calculate the distribution of laminar lengths. The length of the laminar bursts, i.e. the number of consecutive sites which follow periodic behaviour before being interrupted by chaotic bursts is calculated. The distribution for this length shows a power law behaviour with $P(l) \approx l^\zeta$. This shows the presence of long-range spatial correlations in the lattice. The power law exponent characterises the type of intermittency. The distribution of laminar lengths for three kinds of intermittency is shown in Fig.2. The exponent ζ_1 is associated with the tangent bifurcation, the exponent ζ_2 with the Tangent-period-doubling bifurcation and period-doubling (sub-critical) and the exponent ζ_3 with tangent-Hopf and the Double Hopf bifurcations. Such power-laws have been observed in various experimental studies of spatio-temporal intermittency [5]. The values of the power-laws are listed in Table I. It is interesting to note that the values seen by us are in reasonable agreement with the power-laws seen in several experiments which involve quasi-one dimensional geometries [16], [17], [18]. The exponent ζ_1 which takes values between [1.9 – 2.2] for our system, is in good agreement with the laminar exponent seen in the case of Rayleigh-Benard convection in an annulus [16], whereas the exponent ζ_3 which takes the values in the range [0.61 – 0.72] agrees well with the laminar exponent seen in the case of the roll coating system [17]. The exponent ζ_2 which takes values in the range [1.3 – 1.35] and which is seen in the case of the periodic structure with long range spatial correlations is a lower bound on the laminar exponents seen in the case of the Rayleigh-Benard convection seen in a channel and in the Taylor Dean system where spatio-temporal intermittency is seen [20], [18]. Our numerical studies show that this is due to the fact that the temporally periodic structure with long-range spatial correlations undergoes a further bifurcation to spatio-temporal intermittency in the neighbourhood of $\epsilon = 0.368, \gamma = 0.56$ for bifurcations from the synchronised fixed point $x = 0.0$. The distribution of laminar lengths in this region shows power-law behaviour with an exponent $\zeta_G \approx 1.63$ which is in very good agreement with the laminar exponents observed for convection in a channel and for the Taylor-Dean system. This bifurcation does not appear to be local in nature. Details of this bifurcation are being explored further.

All the results above are obtained for bifurcations from the synchronised fixed points for a lattice where the map parameter takes two distinct values one at each alternate site.

These results can be easily generalised to higher spatial and temporal periods and to lattices where the inhomogeneity has different periodicities. Bifurcations from higher spatial temporal periods can also give rise to spatio-temporal or spatial/temporal intermittency in many additional regions of parameter space. The question of whether the spatio-temporal intermittency with or without the accompanying power-laws for the distribution of laminar lengths persists in other regions of parameter space is being investigated further [19].

Thus we have shown that both spatial and spatio-temporal intermittency can arise in a inhomogeneous coupled map lattice. The phenomenon of intermittency is more widespread in inhomogeneous lattices than in the case of ordered lattices. The presence of pure spatial intermittency accompanied by temporally periodic behaviour is an interesting feature which arises in the case of the inhomogeneous system. The intermittency arises as a result of bifurcations of co-dimension 2. Such bifurcations are also of interest in the case of other spatially extended systems [15]. The distributions of laminar lengths exhibit three distinct kinds of power laws, each associated with a distinct kind of bifurcation. The structure associated with the tangent-period-doubling bifurcation undergoes a further bifurcation associated with an additional power-law. The values obtained for these power-laws are in reasonable and intriguing agreement with those observed in a variety of experiments involving quasi-one dimensional geometries. We hence hope our analysis will be useful for the understanding of intermittent phenomena arising in other spatially extended systems, and in discussions of their genericity and universality.

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TABLE 1.1

Fixed Point $x = 0$

Type	Parameter region	Eigenvalue	$P(l) \approx l^\zeta$	Nature
Tangent Bifurcation	$\epsilon = \frac{(5\gamma-21)}{(9\gamma-40)}, \frac{(3\gamma-9)}{(7\gamma-24)}$	+1	ζ_1	ST
Tangent-PD Bifurcation	$\epsilon = 0.39, \gamma = 0.56$	+1, -1	ζ_2	Spatial
Tangent-Hopf Bifurcation	$\epsilon = 0.245, \gamma = 2.87$	$a \pm ib, +1$	ζ_3	ST

TABLE 1.2

Fixed Point $x = \frac{\mu-\gamma\epsilon-1}{\mu-\gamma\epsilon}$

Type	Parameter region	Eigenvalue	$P(l) \approx l^\zeta$	Nature
Tangent Bifurcation	$\epsilon = \frac{5\gamma+8 \pm (5\gamma-192\gamma+64)^{\frac{1}{2}}}{8\gamma}$	+1	ζ_1	ST
P D Bifurcation	$\epsilon = \frac{\gamma^2-3\gamma-8 \pm ((\gamma^2-3\gamma-8)^2-8\gamma(\gamma-4)(\gamma-6))^{\frac{1}{2}}}{(2\gamma(\gamma-4))}$	+1	ζ_2	ST
Tangent - PD Bifurcation	$\epsilon = 0.63, \gamma = 1.180$	-1, +1	ζ_2	Spatial
Tangent-Hopf Bifurcation	$\epsilon = 0.06, \gamma = 2.9801$	$+1, a \pm ib$	ζ_3	ST
Double-Hopf Bifurcation	$\epsilon = 0.5, \gamma = 2$	$a \pm ib, c \pm id$	ζ_3	ST

TABLE 2

Experiment	Value of laminar scaling exponent	CML exponent ζ	Range of ζ
Convection in an annulus	1.9 ± 0.1	ζ_1	$1.9 - 2.2$
Roll coating system	0.63 ± 0.02	ζ_3	$0.61 - 0.72$
Convection (channel)	1.6 ± 0.2	ζ_G	$1.62 - 1.65$
Taylor Dean system	1.67 ± 0.14	ζ_G	$1.62 - 1.65$

I. FIGURE CAPTIONS

1. **Fig 1(a)** shows the spatial intermittency in the lattice associated with tangent - period doubling bifurcation. i refers to position of site on the lattice.**Fig 1(b)** shows two iterates of the lattice. As can be seen both regular and irregular sites behave periodically in time.The fixed point is $\xi = \frac{\gamma\epsilon-\mu-2}{\mu-\gamma\epsilon}$
2. **Fig 2** The scaling behaviour of the laminar sites is shown. $P(l)$ is the probability of obtaining a laminar region of length l . The length of the laminar region is defined as number of adjacent sites which remain within a particular accuracy. Three different exponents are found corresponding to three kinds of intermittency.The fixed point is $\xi = \frac{\gamma\epsilon-\mu-2}{\mu-\gamma\epsilon}$.The behaviour observed was obtained for a lattice of size 50,000 iterated for 20000 iterates starting with 100 random initial conditions. The accuracy to which the laminarity of the region was checked was 10^{-5} . The asterisks denote the behaviour with exponent ζ_1 , crosses ζ_2 and pluses denote ζ_3 . The ranges of the exponents are given in the caption of Table I. ζ_3 shows departures from this power-law over the third decade. The axes are marked in the natural log-scale.
3. **Fig 3** The spatial second return map $x(i+2)$ vs $x(i)$ where i is the site index is plotted for $\xi = \frac{\gamma\epsilon-\mu-2}{\mu-\gamma\epsilon}$.

II. TABLE CAPTIONS

1. **Table 1.1** We list the bifurcations from the synchronised fixed point $x = 0$. The type of bifurcation involved, the region in parameter space where the bifurcation conditions are satisfied, the manner in which the eigenvalue crosses unit circle and the power law exponent $P(l) \approx l^{\zeta_i}$ are listed. The range of the exponent ζ_1 is [1.9 – 2.2], ζ_2 lies in the range [1.3 – 1.35] and ζ_3 in the range [0.61 – 0.72]. We also identify the nature of the intermittency, whether spatial or spatio-temporal(ST). In **Table 1.2** the same quantities are given for the other fixed point.
2. **Table 2** This table lists the values of the spatial laminar exponent observed in experiments (values as quoted in Ref. [5]) and compares their values with the laminar exponents of our CML model. It is clear that the exponents ζ_1 and ζ_3 are directly observed in fluid experiments on convection in an annulus and in the roll coating system. As explained in the text, the exponent ζ_2 serves as a lower bound on the exponent observed in convection in a channel and in the Taylor Dean system. The experimentally observed exponent actually coincides with the CML exponent ζ_G which occurs when the structure associated with the exponent ζ_2 undergoes a further bifurcation.

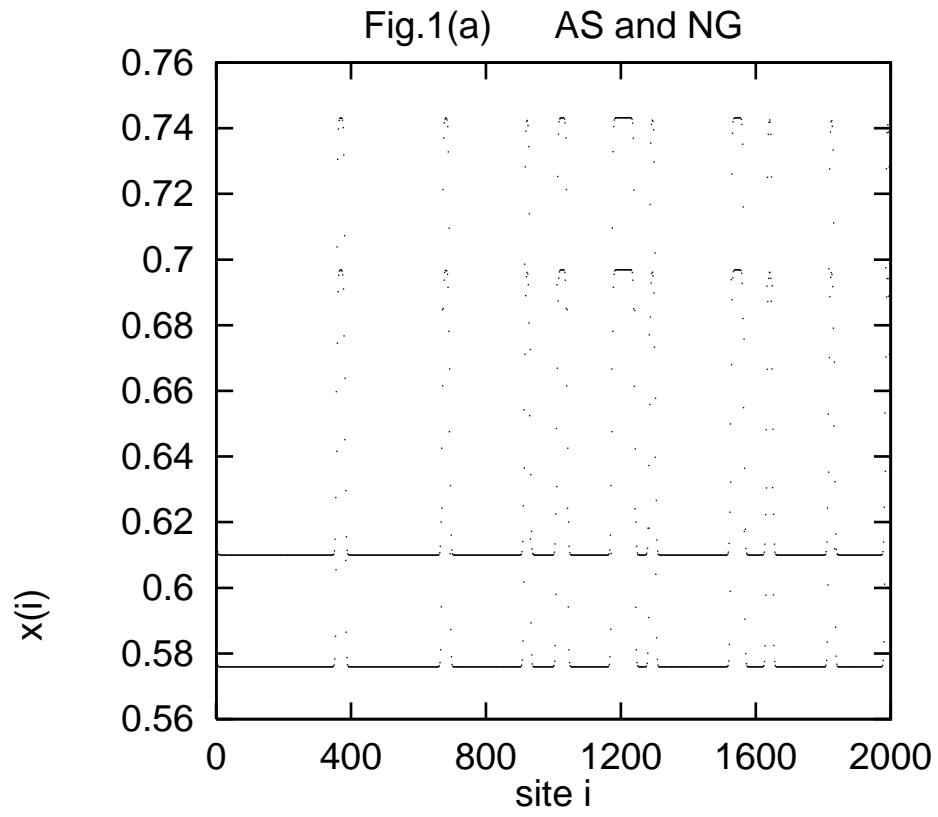


Fig 1b AS and NG

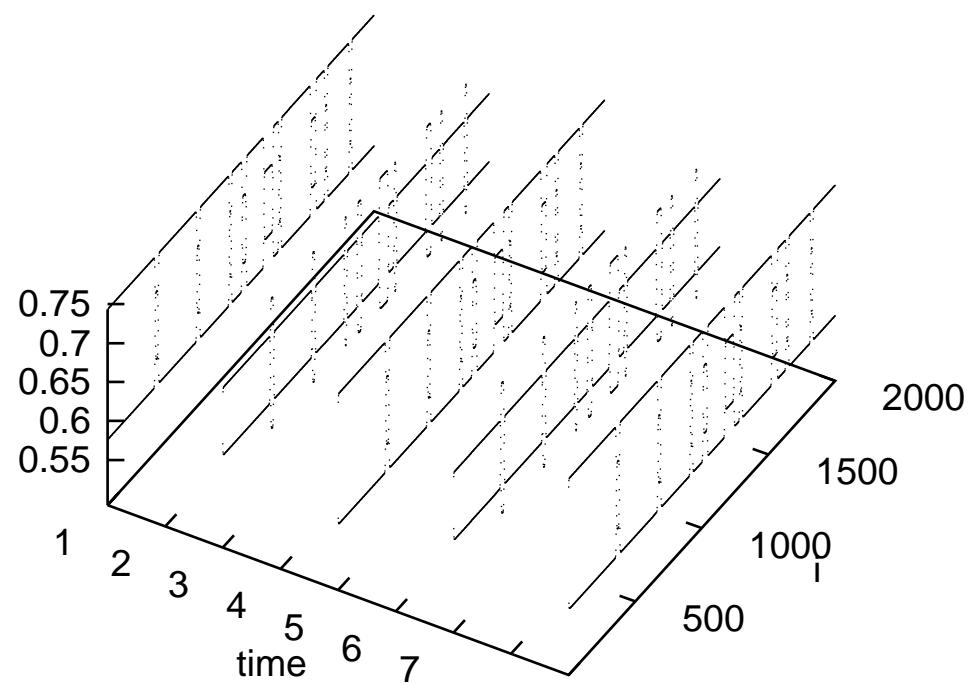


Fig.2

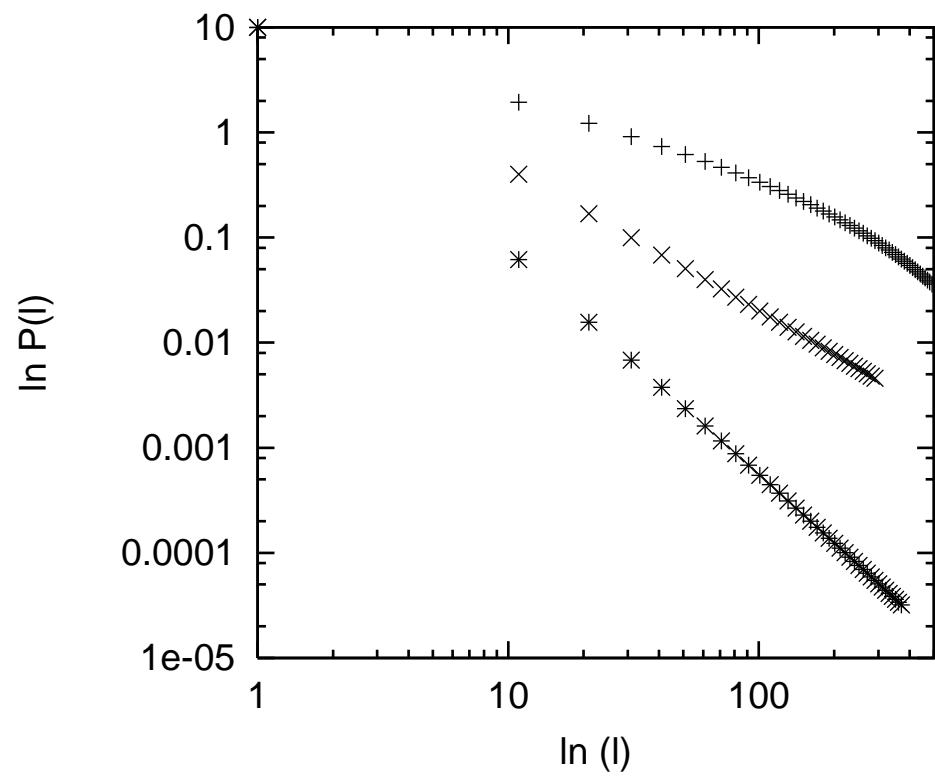


Fig 4 AS and NG

